Hopf-bifurcations on a Nonlinear Chaotic Discrete Model

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Abstract: In this paper we highlight some analytical and numerical discussion of Hopf bifurcation for the nonlinear two-dimensional chaotic map in the plane \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) given by \( f(x, y) = (a - bx + x^2 y, bx + y - x^2 y) \) where the adjustable parameters \( a, b \in \mathbb{R} \).

Here we firstly show that if the nonlinear map \( f \) undergoes subcritical Hopf bifurcation, then \( f^2 \) undergoes critical Hopf bifurcation. Secondly, we show that our numerical and graphical investigations have established some fascinating observation between Hopf bifurcation and Period-doubling bifurcation.

Key Words: Supercritical Hopf bifurcation / Subcritical Hopf bifurcation / Period-doubling bifurcation / Nonlinear / Chaotic.

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1. INTRODUCTION

In case of one-dimensional maps, the lack of hyperbolicity is usually a signal for the occurrence of bifurcations. In these systems, bifurcations occur when the eigenvalues of a periodic point is either -1 (the saddle node bifurcation) or -1 (the period doubling bifurcation). For higher dimensional systems, these types of bifurcations also occur, but there are other possible bifurcations of periodic points as well. The most typical of these is the Hopf bifurcation. Although this type of bifurcation was known and understood by the great French mathematician Jules Henri Poincaré as well as the Soviet mathematician Aleksandr Andronov, but Eberhard Hopf (the German mathematician) was the first to extend these ideas to higher dimensional state spaces.

In the theory of bifurcations, a Hopf bifurcation refers to the local birth and death of a periodic solution as a pair of complex conjugate eigenvalues of the linearization around the fixed point which crosses the imaginary axis of the complex plane as the parameter varies. Under reasonably generic assumptions about the dynamical system, we can expect to see a small amplitude limit cycle branching from the fixed point [1-3, 5-8].

We now highlight some useful concepts which are absolutely useful for our purpose.

1.1 DISCRETE DYNAMICAL SYSTEMS

Any \( C^k (k \geq 1) \) map \( E : U \rightarrow \mathbb{R}^n \) on the open set \( U \subset \mathbb{R}^n \) defines an n-dimensional discrete-time (autonomous) smooth dynamical system by the state equation \( x_{t+1} = E(x_t), t = 1, 2, 3, \ldots \), where \( x_t \in \mathbb{R}^n \) is the state of the system at time \( t \) and \( E \) maps \( x_t \) to the next state \( x_{t+1} \).

Starting with an initial data \( x_0 \), repeated applications (iterates) of \( E \) generate a discrete set of points (the orbits) \( \{E^t(x_0): t = 0, 1, 2, 3, \ldots \} \), where \( E^t(x) = E \circ E \circ \ldots \circ E(x) \) [9].

1.2 Bifurcation

Bifurcation, as a scientific terminology, has been used to describe significant and qualitative changes that occur in the solution curves of a dynamical system, as the key system parameters are varied. Very frequently, it is used to describe the qualitative stability changes of the solution curves of a nonlinear dynamical system [7]. Many dynamical systems depend on parameters. Normally a gradual variation of a parameter in the system corresponds to the gradual variation of the solutions of the problem. However, there exists a large number of problems for which the number of solutions changes abruptly and the structures of solution manifolds vary dramatically when a parameter passes through some critical values. This kind of phenomenon is called bifurcation and these parameter values are called bifurcation values (or bifurcation points). In the case of a diffeomorphism \( E \), period-doubling bifurcations occur when one of the eigenvalues of the derivative \( DE^n(x) \) = -1.

Bifurcation theory is a method for studying how solutions of a nonlinear problem and their stability changes as the parameter vary. The onset of chaos is often studied by bifurcation theory. For example, in parameterized families of one-dimensional map, chaos develops via period-doubling cascade [4].

1.3 The Hopf bifurcation theorem for maps in the plane \( \mathbb{R}^2 \)

Let \( E_b(x, b) \), where \( b \) is the bifurcation parameter, be a one-parameter family of maps in the plane \( \mathbb{R}^2 \) satisfying the following conditions:

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127
(C1) An isolated fixed point \( \mathbf{x}^* (b) \) exists,

(C2) The map \( E_b \) is \( C^k \) \((k \geq 3)\) in the neighborhood of \((\mathbf{x}^* (b_0); b_0)\).

(C3) The Jacobian \( D\mathbf{x} E(\mathbf{x}^* (b); b) \) matrix possesses a pair of complex, simple
eigenvalues \( \gamma(b) = \alpha(b) + io(b) \) and \( \overline{\gamma}(b) \), such that at
the critical value \( b = b_0 \)
\[
|\gamma(b_0)| = 1, \quad \frac{d|\gamma(b)|}{db} (b = b_0) > 0,
\]

(C4) \( (\gamma(b_0))^j \neq 1, \quad j = 3, 4. \)

(Existence) Then there exists a real number \( \varepsilon_0 > 0 \) and a
\( C^{k-1} \) function such that
\[
b(\varepsilon) = b_0 + b_1 \varepsilon + b_2 \varepsilon^3 + O(\varepsilon^4)
\]
such that for each \( \varepsilon \in (0, \varepsilon_0] \) the map \( E_b \) has an invariant manifold \( H(b) \), that is,

\[ E(H(b); b) = H(b). \]

The manifold \( H(b) \) is \( C^b \) diffeomorphic to a circle and consists of points at a distance
\[ O(|\varepsilon|^{1/2}) \] of \( \mathbf{x}^* (b) \), for \( b = b(\varepsilon) \).

(Uniqueness) Each compact invariant manifold close to
\( \mathbf{x}^* (b) \) for \( b = b(\varepsilon) \) is contained in \( H(b) \cup \{0\} \).

(Stability) If \( \rho_3 < 0 \) (respectively \( \rho_3 > 0 \)) then for \( r < 0 \)
(respectively \( r > 0 \)), the fixed point \( \mathbf{x}^* (b(\varepsilon)) \) is stable
(respectively unstable) and for \( r > 0 \) (respectively \( r < 0 \)), the
fixed point \( \mathbf{x}^* (b(\varepsilon)) \) is unstable (respectively stable) and the
surrounding manifold \( H(b(\varepsilon)) \) is attracting (respectively repelling). When \( \rho_3 < 0 \) (respectively \( \rho_3 > 0 \)) the bifurcation
at \( b = b(\varepsilon) \) is said to be supercritical (respectively
subcritical) [10].

Armed with all these ideas and concepts, we now proceed
to concentrate to our main aim and objectives.

2. GENERAL THEORY

We consider a discrete-time dynamical system
\[
\Xi \mapsto E(\Xi; b), \quad \Xi \in \mathbb{R}^2
\]
depending on the parameter \( b \in \mathbb{R} \), where \( E \) is smooth. Suppose that for each \( b \) near the origin
there is a fixed point of \( E \), denoted by \( \mathbf{x}^* (b) \), with the
Jacobian matrix \( D\mathbf{x} E(\mathbf{x}^* (b); b) \) having a pair of purely
imaginary (complex conjugate) eigenvalues, denoted by \( \eta(b) \) and \( \overline{\eta}(b) \), cross the imaginary axis in the complex eigenplane
as the parameter \( b \) passes through the origin. The fixed
point is then a center for the linearised system, but for the
nonlinear system the higher order terms will typically convert
it into either an attracting or repelling focus. In the following,
we describe a method for determining which of these two
alternatives occurs; it involves making a series of coordinate
changes to reduce the system to a normal form.

Firstly, by making a linear change of coordinates we can
arrange that \( \mathbf{x}^* (b) = (0, 0) \) and then in complex notation,
\[
z = x + iy
\]
we obtain an expression for the system in the variable \( z \) as
\[
E(z) = \eta z + A_1 z^2 + B_1 z \overline{z} + C_1 z^2 + M_1 z^2 + \ldots
\]
where \( A_1, B_1, C_1, M_1 \) are complex constants.

Secondly, to make a new change of variables to eliminate
the quadratic terms, we put \( w = z + \mu \zeta^2 + \nu \overline{\zeta} + \rho \zeta \overline{\zeta} \). Then we expand \( E(w) \), keeping only terms up to second order (and noting, e.g. that the difference between \( \zeta^2 \) and \( W^2 \) is third order, so \( \zeta^2 \) can be replaced by \( W^2 \), etc.)
\[
E(z) = E(\zeta) + 2\mu E(z) \zeta + 2 \rho \overline{E}(\zeta)\overline{z}
\]
\[
= \eta \zeta + A_1 \zeta^2 + B_1 \zeta \overline{z} + C_1 \zeta^2 + M_1 \zeta^2
\]
Again we have
\[
\eta w = \eta \zeta + \mu \zeta^2 + \nu \zeta \overline{\zeta} + \rho \zeta \overline{\zeta}^2
\]
and equating the above expressions for \( E(w) \) and \( \eta z \) we find that, if we take
\[
\mu = -A_1 / \eta, \quad \nu = -B_1 / \overline{\eta}, \quad \rho = C_1 / (\eta - 2 \overline{\eta})
\]
then the system reduces to \( E(w) = \eta w + \) terms of 3\textsuperscript{rd} and higher order. Thus we can write
\[
E(w) = \eta w + p w^3 + q w^2 \overline{w} + r \zeta \overline{w}^2 + s \zeta \overline{w}^3 + \ldots
\]
where \( p, q, r, s \) are complex numbers.

Thirdly, we eliminate as many of the third order terms as
possible (by a similar procedure to that used in second step).
We make a change of variable
\[
t = w + \alpha \zeta^2 + \beta \zeta \overline{\zeta} + \gamma \zeta \overline{\zeta}^2 + \delta \zeta \overline{\zeta}^3
\]
and choose \( d, e, j, l \) so as to eliminate the third order terms.

www.ijsea.com 128
Under these situations, the system reduces to the normal form as

\[ E(r) = \eta r (1 + H[I]^2 + O[I]^4) \]

where,

\[ H = \frac{(1 - 2\eta)A_1B_1}{\eta^2(\eta - 1)} + \frac{B_1\tilde{B}_1}{\eta - 1} + \frac{2C_1\tilde{C}_1}{\eta^2 - 1} + \eta^{-1}M_1 \]

Interestingly, Supercritical (respectively Subcritical) Hopf bifurcations occurs when \( H < 0 \) (respectively \( H > 0 \)). If \( H = 0 \), we need to consider higher order terms to draw the conclusion.

3. OUR MAIN ANALYTICAL DISCUSSION

To study the Hopf bifurcation in a concrete way, we now consider a nonlinear two-dimensional map and make the necessary analytic deduction to study their intrinsic properties. Our 2-D map is

\[ f(x, y) = (a - bx + x^2y, bx + y - x^2y) \] (1.1)

where the parameter \( b \in \mathbb{R} \), and the parameter \( a \) is fixed as \( a = 1 \).

Let \((x_0, y_0)\) be a fixed point of the map \( f \). Let us change to a new coordinate \((\xi, \eta)\) by the relation \( \xi = x - x_0, \eta = y - y_0 \) so that \((0,0)\) is a fixed point of the map \( f \). Now \( x = \xi + x_0, y = \eta + y_0 \). Then first iteration of the system (1.1) becomes

\[
\begin{align*}
\dot{x}_1 &= 1 - b(x + x_0) + (x + x_0)^2(\eta + y_0) - x_0 \\
\dot{y}_1 &= b(x + x_0) + (\eta + y_0) - (x + x_0)^2(\eta + y_0) - y_0 \\
\dot{x}_1 &= (1 - x_0 - bx_0 + x_0^2y_0)(x + x_0)^2 + 2x_0y_0^2 + 2x_0x_0^2 + x_0^2y_0^2 \\
\dot{y}_1 &= (b(1 - x_0)(x + x_0) + (b - 2x_0)y_0)(x + x_0)^2 + (1 - x_0)^2 + x_0^2y_0^2
\end{align*}
\]

For our convenience, we write \( x = \xi, y = \eta \).

So

\[
\begin{align*}
\dot{x}_1 &= m_1 + n_1x + p_1y + q_1x^2 + \eta_1xy + s_1y^2 + t_1x^3 + u_1x^2y \\
&\quad + \eta_1xy^2 + w_1y^3 \\
\dot{y}_1 &= M_1 + N_1x + P_1y + Q_1x^2 + R_1xy + S_1y^2 + T_1x^3 + U_1x^2y + V_1xy^2 + W_1y^3
\end{align*}
\] (1.2)

where

\[
m_1 = 1 - x_0 - bx_0 + x_0^2y_0, m_1 = -b + 2x_0y_0, p_1 = x_0^2, \\
q_1 = y_0, n_1 = 2x_0, q_1 = 0, q_1 = 0, q_1 = 1, \eta_1 = 0, w_1 = 0, \\
M_1 = bx_0 - x_0^2y_0, N_1 = b - 2x_0y_0, R_1 = 1 - x_0^2, Q_1 = -y_0, \\
R_1 = -2x_0, S_1 = 0, T_1 = 0, U_1 = -1, V_1 = 0, W_1 = 0.
\]

We now iterate the system (1.2). Let \( f(\dot{x}_1, \dot{y}_1) = (\dot{x}_2, \dot{y}_2) \), where

\[
\begin{align*}
\dot{x}_2 &= m_2 + n_2(m_1 + n_1x + p_1y + q_1x^2 + \eta_1xy + s_1y^2 + t_1x^3 + u_1x^2y) + \eta_1xy^2 + w_1y^3 + \eta_1xy^2 + w_1y^3 \\
&\quad + (1 - x_0 - bx_0 + x_0^2y_0)(x + x_0)^2 + 2x_0y_0^2 + 2x_0x_0^2 + x_0^2y_0^2 \\
&\quad + (b(1 - x_0)(x + x_0) + (b - 2x_0)y_0)(x + x_0)^2 + (1 - x_0)^2 + x_0^2y_0^2 \\
\dot{y}_2 &= M_2 + N_2x + P_2y + Q_1x^2 + R_1xy + S_1y^2 + T_1x^3 + U_1x^2y + V_1xy^2 + W_1y^3 \\
&\quad + (1 - x_0 - bx_0 + x_0^2y_0)(x + x_0)^2 + 2x_0y_0^2 + 2x_0x_0^2 + x_0^2y_0^2 \\
&\quad + (b(1 - x_0)(x + x_0) + (b - 2x_0)y_0)(x + x_0)^2 + (1 - x_0)^2 + x_0^2y_0^2
\end{align*}
\]

After simplification we find

\[
\begin{align*}
\dot{x}_2 &= (m_2 + n_2m_1 + m_2P_1 + m_2^2q_1 + m_2M_1q_1 + m_2^2M_1q_1) + (n_2^2 + N_1p_1 + 2m_2n_2q_1 + m_2N_1q_1 + m_2N_1q_1 + 2m_2M_1n_2q_1 + m_2^2N_1q_1) \\
&\quad + (n_2^2 + N_1p_1 + 2m_2n_2q_1 + m_2N_1q_1 + m_2N_1q_1 + 2m_2M_1n_2q_1 + m_2^2N_1q_1 + 2m_2M_1n_2q_1 + m_2^2N_1q_1) \\
&\quad + 2m_2M_1n_2q_1 + m_2^2Q_1) \dot{x}_2 + (n_2^2 + N_1p_1 + 2m_2n_2q_1 + m_2N_1q_1 + m_2N_1q_1 + 2m_2M_1n_2q_1 + m_2^2N_1q_1) \dot{y}_2 \\
&\quad + (n_2^2 + N_1p_1 + 2m_2n_2q_1 + m_2N_1q_1 + m_2N_1q_1 + 2m_2M_1n_2q_1 + m_2^2N_1q_1) \dot{y}_2 \\
&\quad + (n_2^2 + N_1p_1 + 2m_2n_2q_1 + m_2N_1q_1 + m_2N_1q_1 + 2m_2M_1n_2q_1 + m_2^2N_1q_1) \dot{y}_2
\end{align*}
\]

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\( y_2 = (M_1 + m_1 N_1 + M_1 R_1 + m_1^2 q_1 + m_1 M_1 R_1 + m_1^2 M_1 U_1) + (n_1 N_1 + N_1 R_1 + 2m_1 q_1 R_1 + M_1 p_1 R_1 + n_1 N_1 R_1 + 2m_1 q_1 R_1 + + M_1 p_1 R_1 + 2m_1 q_1 U_1 + + 2m_1 q_1 U_1 + M_1 p_1 R_1 + 2m_1 q_1 U_1 + + (N_1 q_1 + + n_1^2 q_1 + 2m_1^2 q_1^2 + + P_1 q_1 + n_1 N_1 q_1 + + M_1 q_1 R_1 + + M_1 q_1 U_1 + + 2m_1 q_1 U_1 + + M_1 q_1 U_1) x + + (N_1 q_1 + + n_1^2 q_1 + 2m_1^2 q_1^2 + + P_1 q_1 + n_1 N_1 q_1 + + M_1 q_1 R_1 + + M_1 q_1 U_1 + + 2m_1 q_1 U_1 + + M_1 q_1 U_1) y + + (N_1 q_1 + + n_1^2 q_1 + 2m_1^2 q_1^2 + + P_1 q_1 + n_1 N_1 q_1 + + M_1 q_1 R_1 + + M_1 q_1 U_1 + + 2m_1 q_1 U_1 + + M_1 q_1 U_1) \)

Thus
\[
\begin{aligned}
\dot{x}_2 &= m_2 x + p_2 x + q_2 x^2 + r_2 x y + r_2 x^2 + t_2 x^3 + \\
&= u_2 x^2 y + v_2 x^2 + w_2 x^2 + y \cdot 2 x^3 + \\
\dot{y}_2 &= M_2 + N_2 + P_2 + Q_2 x^2 + R_2 x y + S_2 y^2 + T_2 x^3 + \\
&= U_2 x^2 y + V_2 x^2 + W_2 y^2 + \\
\end{aligned}
\]

where \( m_2, m_2, p_2, \ldots, M_2, N_2, P_2, \ldots \) etc. can be determined from above.

Proceeding in the same manner we can determine the expressions for any number of iterations of the map \( f \).

### 4. METHOD FOR THE COMPUTATION OF HOPF BIFURCATION POINTS

We now develop here the eigenvalue theory of the Jacobian matrix of the vector field \( f \). Suppose
\[
M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]
as the Jacobian matrix of \( f \), then the eigenvalues \( \eta \) of \( M \) are given by
\[
\eta = \frac{1}{2}(A + D) \pm \sqrt{(A - D)^2 + 4BC} = \phi \pm i\psi \quad \text{(say)}
\]
(Here the patial derivatives \( A, B, C \) and \( D \) are taken so that that the eigenvalues \( \eta \) come out as complex numbers)

If \( \varepsilon = \begin{bmatrix} \mu \\ \nu \end{bmatrix} \) is the eigenvector of the transpose of \( M \) corresponding to the eigenvalue \( \eta \), then
\[
\begin{bmatrix} A - \eta & B \\ C & D - \eta \end{bmatrix} \begin{bmatrix} \mu \\ \nu \end{bmatrix} = 0 \Rightarrow \mu = C, \nu = \eta - A
\]
\[= (\phi + i\psi) - A = (\phi - A) + i\psi \]

Let us put \( z = \mu x + \nu y \). This implies \( \bar{z} = \bar{\mu}x + \bar{\nu}y \). Consequently
\[
x = \frac{\bar{z} - \nu T}{\bar{\nu} - \bar{\mu} T}, \quad y = \frac{\bar{z} - \mu T}{\bar{\nu} - \bar{\mu} T}
\]
\[
\Rightarrow x = \frac{\bar{z} - \nu T}{\bar{\nu} - \bar{\mu} T}, \quad y = \frac{\bar{z} - \mu T}{\bar{\nu} - \bar{\mu} T}
\]
where \( \bar{\sigma} = \bar{\nu} - \bar{\mu} T = C(\bar{v} - \nu) = -2i\psi C \) and
\[
\psi = \bar{\nu} - \bar{\mu} T = -C = 2i\psi C
\]
\[
\Rightarrow \bar{z} = c \begin{bmatrix} \mu & \nu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{where} \ x = \begin{bmatrix} x \\ y \end{bmatrix}
\]

Now
\[
= \mu x + \nu y
\]
\[= \mu(q x^2 + rxy + sy^2 + txy + u^2 + vxy + wxy + v'x^2 + w') \quad \text{(say)}
\]
\[
= Q x^2 + R x y + S y^2 + T x^3 + U y^2 + V x y^2 + W x y^3
\]

(For simplicity, we omit the lower suffixes from \( q, r, \ldots, Q, R, \ldots \) etc. Also to evaluate \( H \) we do not need the coefficients of linear terms and hence the linear terms are omitted.)
\[
= (\mu q + \nu Q)x^2 + (\mu r + \nu R)x y + (\mu s + \nu S)y^2 + (\mu t + \nu T)x^3 + (\mu u + \nu U)x^2 y + (\mu v + \nu V)x y^2 + (\mu w + \nu W)y^3
\]
\[
= Q x^2 + R x y + S y^2 + T x^3 + U y^2 + V x y^2 + W x y^3
\]

\[
= Q \left( \frac{\bar{v} - \nu T}{\bar{\nu} - \bar{\mu} T} \right)^2 + R \left( \frac{\bar{v} - \nu T}{\bar{\nu} - \bar{\mu} T} \right)^3 + S \left( \frac{\bar{v} - \mu T}{\bar{\nu} - \bar{\mu} T} \right)^2 + \\
= T \left( \frac{\bar{v} - \nu T}{\bar{\nu} - \bar{\mu} T} \right)^3 + U \left( \frac{\bar{v} - \nu T}{\bar{\nu} - \bar{\mu} T} \right)^3 + V \left( \frac{\bar{v} - \mu T}{\bar{\nu} - \bar{\mu} T} \right)^2 + \\
= \left( \frac{\bar{v} - \mu T}{\bar{\nu} - \bar{\mu} T} \right)^2 + W \left( \frac{\bar{v} - \mu T}{\bar{\nu} - \bar{\mu} T} \right)^3 + \\
= e_1 + i e_2 \quad \text{(say)}
\]

\[
Q^2 = (\mu q + \nu Q)(\phi - A - i\psi)^2
\]
\[
= (\mu q + (\phi - A)Q)((\phi - A)^2 - \psi^2) + 2\psi^2 Q(\phi - A) + \\
= \left[ -2(\phi - A)\psi \right] Q(\phi - A)^2 - \psi^2 + \psi Q((\phi - A)^2 - \psi^2) + \\
= e_1 + i e_2 \quad \text{(say)}
\]
\[
\begin{align*}
\bar{R}\bar{v} &= \mu(\mu v + vR) \bar{v} \\
&= \left[ \mu^2 v + \mu R(\phi - A) \right] (\phi - A) + \mu R
\end{align*}
\]
\[
\begin{align*}
\bar{S} \bar{\mu}^2 &= (\mu + s) \mu^2 = (\mu^3 + s^2 S(\phi - A) + iS \psi) \mu^2 \\
&= \mu^3 + s^2 S(\phi - A) + iS \psi = R_{\text{gs}} + iR_{\text{gs}} \quad \text{(say)}
\end{align*}
\]

Taking
\[
A_0 = \left[ \frac{Q^2}{\omega^2} + \frac{R^2 \bar{v} \bar{v}}{\omega^2} + \frac{S^2 \bar{\mu}^2}{\omega^2} \right]
\]
\[
= \frac{1}{4V^2} \left[ (e_1 - e_3 + e_5) + i(e_2 - e_4 + e_6) \right]
\]
\[
= G_0 + iH_0 \quad \text{(say)}
\]

Again
\[
\begin{align*}
\bar{R} \bar{v} &= \mu \bar{R} (\bar{v} + v) = -\mu (\mu v + vR) (\bar{v} + v) \\
&= 2(\mu v + \mu R(\phi - A)) (\phi - A) + i(2\mu R \psi(\phi - A)) \\
&= f_3 + i\eta \quad \text{(say)}
\end{align*}
\]
\[
\begin{align*}
2S \bar{\mu} &= 2\mu^2 (\mu v + vS) \\
&= 2\mu^2 (\mu S(\phi - A) + iS^2 \psi) = f_3 + i\eta \quad \text{(say)}
\end{align*}
\]
\[
B_1 = \left[ \frac{2Q^2}{\omega^2} - \frac{R (\bar{v} \bar{v} + v^2)}{\omega^2} - \frac{2S \bar{\mu}^2}{\omega^2} \right]
\]
\[
= \frac{1}{4V^2} \left[ (f_1 - f_3 + f_5) + i(f_2 - f_4 + f_6) \right]
\]
\[
= I_0 + iI_0 \quad \text{(say)}
\]

Also
\[
Q^2 = [I(\mu v + (\phi - A)Q)] (\phi - A) - 2 \mu \psi Q(\phi - A) + i[I(\mu v + (\phi - A)Q)] [2\mu v(\phi - A) + \mu Q(\phi - A)] \
= \psi \mu (\mu v + R(\phi - A)) + i\psi R(\phi - A) \\
= \psi \mu + ig_3 \\
\]
\[
S^2 \mu^2 = (\mu^3 + s^2 S(\phi - A) + iS \psi S) \\
= g_3 + ig_3 \quad \text{(say)}
\]

After calculating above somewhat complicated expressions, we have the stability index \( H \) as
\[
H = \frac{(1 - 2\eta)A_1 B_1}{\eta^3 - \eta^2} + \frac{\bar{B_1} \bar{B_1}}{\eta^3 - \eta^2} + \frac{2C_1 \bar{C_1}}{\eta^3 - \eta^2} + \eta^{-1}M_1
\]
\[
= A_1 + iA_2 \quad \text{(say)}
\]
\[
\Re \left( \frac{(1 - 2\eta)A_1 B_1}{\eta^3 - \eta^2} \right) = \Re \left( \frac{1}{\eta^3 - \eta^2} \right) = \frac{A_1}{\eta} + \frac{\bar{A_2}}{\eta^2}
\]
Under this situation, we have

\[ \text{Re } H = X_1^{00} + X_2^{00} + X_3^{00} + X_4^{00} \]

Analogous theory can be developed for any number of iterations of the map \( f \). To evaluate the values of \( H \) we compose a computer program and obtain the values of \( H \) for the maps \( f \) and \( f^2 \) as tabulated below in tables 1.1 and 1.2.

<table>
<thead>
<tr>
<th>( PV )</th>
<th>( REV )</th>
<th>( IEV )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.420000</td>
<td>0.210000</td>
<td>0.681249</td>
<td>-8.711063</td>
</tr>
<tr>
<td>0.440000</td>
<td>0.220000</td>
<td>0.698856</td>
<td>-8.197007</td>
</tr>
<tr>
<td>0.460000</td>
<td>0.230000</td>
<td>0.716170</td>
<td>-7.744707</td>
</tr>
<tr>
<td>0.480000</td>
<td>0.240000</td>
<td>0.733212</td>
<td>-7.342201</td>
</tr>
<tr>
<td>0.500000</td>
<td>0.250000</td>
<td>0.750000</td>
<td>-6.978884</td>
</tr>
<tr>
<td>0.520000</td>
<td>0.260000</td>
<td>0.766551</td>
<td>-6.644871</td>
</tr>
<tr>
<td>0.540000</td>
<td>0.270000</td>
<td>0.782879</td>
<td>-6.330438</td>
</tr>
<tr>
<td>0.560000</td>
<td>0.280000</td>
<td>0.798999</td>
<td>-6.025507</td>
</tr>
<tr>
<td>0.580000</td>
<td>0.290000</td>
<td>0.814923</td>
<td>-5.719211</td>
</tr>
<tr>
<td>0.600000</td>
<td>0.300000</td>
<td>0.830662</td>
<td>-5.399610</td>
</tr>
<tr>
<td>0.620000</td>
<td>0.310000</td>
<td>0.846227</td>
<td>-5.053740</td>
</tr>
<tr>
<td>0.640000</td>
<td>0.320000</td>
<td>0.861626</td>
<td>-4.668285</td>
</tr>
<tr>
<td>0.660000</td>
<td>0.330000</td>
<td>0.876869</td>
<td>-4.231255</td>
</tr>
<tr>
<td>0.680000</td>
<td>0.340000</td>
<td>0.891964</td>
<td>-3.734987</td>
</tr>
<tr>
<td>0.700000</td>
<td>0.350000</td>
<td>0.906918</td>
<td>-3.180244</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( PV )</th>
<th>( REV )</th>
<th>( IEV )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.840000</td>
<td>35.547664</td>
<td>35.411240</td>
<td>0.872926</td>
</tr>
<tr>
<td>-0.850000</td>
<td>35.645886</td>
<td>35.509420</td>
<td>0.873715</td>
</tr>
<tr>
<td>-0.855000</td>
<td>35.744326</td>
<td>35.607817</td>
<td>0.874490</td>
</tr>
<tr>
<td>-0.860000</td>
<td>35.842983</td>
<td>35.706433</td>
<td>0.875251</td>
</tr>
<tr>
<td>-0.865000</td>
<td>35.941858</td>
<td>35.805266</td>
<td>0.875998</td>
</tr>
<tr>
<td>-0.870000</td>
<td>36.040951</td>
<td>35.904318</td>
<td>0.876731</td>
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<tr>
<td>-0.875000</td>
<td>36.140262</td>
<td>36.003589</td>
<td>0.877451</td>
</tr>
<tr>
<td>-0.880000</td>
<td>36.239791</td>
<td>36.103077</td>
<td>0.878157</td>
</tr>
<tr>
<td>-0.885000</td>
<td>36.339537</td>
<td>36.202784</td>
<td>0.878850</td>
</tr>
<tr>
<td>-0.890000</td>
<td>36.439502</td>
<td>36.302710</td>
<td>0.879529</td>
</tr>
<tr>
<td>-0.895000</td>
<td>36.539684</td>
<td>36.402853</td>
<td>0.880195</td>
</tr>
<tr>
<td>-0.900000</td>
<td>36.640085</td>
<td>36.503216</td>
<td>0.880848</td>
</tr>
<tr>
<td>-0.905000</td>
<td>36.740704</td>
<td>36.603797</td>
<td>0.881488</td>
</tr>
<tr>
<td>-0.910000</td>
<td>36.841541</td>
<td>36.704597</td>
<td>0.882115</td>
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<tr>
<td>-0.915000</td>
<td>36.942597</td>
<td>36.805615</td>
<td>0.882728</td>
</tr>
<tr>
<td>-0.920000</td>
<td>37.043871</td>
<td>36.906853</td>
<td>0.883330</td>
</tr>
<tr>
<td>-0.925000</td>
<td>37.145363</td>
<td>37.008309</td>
<td>0.883918</td>
</tr>
<tr>
<td>-0.930000</td>
<td>37.247074</td>
<td>37.109984</td>
<td>0.884494</td>
</tr>
</tbody>
</table>

Table 1.2 (For the map \( f^2 \))

<table>
<thead>
<tr>
<th>( PV )</th>
<th>( REV )</th>
<th>( IEV )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.720000</td>
<td>0.360000</td>
<td>0.921737</td>
<td>-2.580188</td>
</tr>
<tr>
<td>-0.740000</td>
<td>0.370000</td>
<td>0.936429</td>
<td>-1.961760</td>
</tr>
<tr>
<td>-0.760000</td>
<td>0.380000</td>
<td>0.950999</td>
<td>-1.362047</td>
</tr>
<tr>
<td>-0.780000</td>
<td>0.390000</td>
<td>0.965453</td>
<td>-0.819575</td>
</tr>
<tr>
<td>-0.800000</td>
<td>0.400000</td>
<td>0.979796</td>
<td>-0.364137</td>
</tr>
<tr>
<td>-0.820000</td>
<td>0.410000</td>
<td>0.994032</td>
<td>-0.010263</td>
</tr>
</tbody>
</table>

In both of these tables \( PV, REV \) and \( IEV \) mean parameter value, \( x \)-coordinate of fixed point, \( y \)-coordinate of fixed point, real part of eigenvalues and imaginary part of eigenvalues respectively.
From the above tables and figures we have found that while the map $f$ undergoes supercritical Hopf bifurcation and $f^2$ undergo subcritical Hopf bifurcation. Furthermore, the graph in the Figure 1.3 shows that there is a close link between Hopf bifurcation and Period-doubling bifurcation.

### 6. Period-doubling and Hopf bifurcation values:

<table>
<thead>
<tr>
<th>Map</th>
<th>Period-doubling bifurcation value</th>
<th>Hopf bifurcation value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>-0.5</td>
<td>-6.978883861237…</td>
</tr>
<tr>
<td>$f^2$</td>
<td>-0.843730536895…</td>
<td>0.873515810209…</td>
</tr>
<tr>
<td>$f^4$</td>
<td>-0.9295533…</td>
<td>-0.000000936344…</td>
</tr>
</tbody>
</table>

### 8. Period-Doubling Tree

*Figure 1.4 The Period-Doubling Tree for the parameter range $-0.945 \leq b \leq -0.4$*

### 9. CONCLUSIONS

The theory described in this paper can be used to determine the Hopf bifurcation values of any nonlinear maps and the behavior of Hopf bifurcation changes from supercritical to subcritical or vice-versa at the period-doubling points.

### REFERENCES


[9] Sandri Marco, *Numerical calculation of Lyapunov Exponents*, University of Verona, Italy